Pismeni ispit 14.7.2015.

1.

pozitronij: e^-e^+

a) $E_n = f(n, K) = ?$

$$E_n = -\frac{K}{n^2}$$

b) $K = ?$

vodikov atom: (e^-p)

$$E_n = -\frac{R}{n^2} \qquad R = k_C^2 \frac{m_e e^4}{2\hbar^2} \approx m_e \Leftarrow \mu = \frac{m_e m_p}{m_e + m_p} = \frac{m_e}{1 + \underbrace{\frac{m_e}{m_p}}_{\ll 1}} \approx m_e!$$

pozitronij: (e^-e^+)

$$E_n = -\frac{K}{n^2} = -\frac{R\frac{\mu'}{m_e}}{n^2} = \left| \mu' = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}! \right| = -\frac{R/2}{n^2}$$

$$\Downarrow$$

$$K = R/2 = \frac{13.6\,\text{eV}}{2} = 6.8\,\text{eV}$$

c) pozitronij → vidljiva svjetlost? (400–700 nm)

$$\left[\frac{1}{\lambda_{mn}} = \frac{R}{hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \right]_{m,n\in\mathbb{N},\ n>m} \text{ vodik} \qquad R_H = 4/b$$

$$\frac{1}{\lambda_{mn}} = \frac{4}{b}\,\frac{n^2 - m^2}{m^2 n^2} \quad \Downarrow \quad \lambda_{mn} = \frac{b}{4}\,\frac{m^2 n^2}{n^2 - m^2}$$

$$\frac{1}{\lambda^{e^+e^-}_{mn}} = \left| R \to K = R/2 \right| = \frac{1}{2}\,\frac{1}{\lambda_{mn}}$$

$b = 364.56$ nm

$b/4 = 91.2$ nm

$$\Rightarrow \lambda^{e^+e^-}_{mn} = 2\lambda_{mn}$$

LYMAN-ova serija $m=1$

$$\lambda^{e^+e^-}_{12} = 2\lambda_{12} = 2\cdot 91.2\,\text{nm}\ \frac{1\cdot 4}{4-1} = 2\cdot 91.2\,\text{nm}\cdot\frac{4}{3} = 2\cdot 121.6\,\text{nm} = 244\,\text{nm}$$

$$\lambda^{e^+e^-}_{1\infty} = 2\lambda_{1\infty} = 2\cdot 91.2\,\text{nm}\ \frac{n^2}{1 - \cancel{\frac{n^2}{\infty^2}}} = 2\cdot 91.2\,\text{mm} = 182.4\,\text{nm} \quad \text{UV}$$

< 400 nm

BALMER-ova serija $m=2$

$$\lambda^{e^+e^-}_{23} = 2\lambda_{23} = 2\cdot 91.2\,\text{nm}\ \frac{4\cdot 9}{9-4} = 2\cdot 91.2\,\text{nm}\,\frac{36}{5} = 2\cdot 657\,\text{nm} = 1314\,\text{nm}$$

$$\lambda^{e^+e^-}_{2\infty} = 2\lambda_{2\infty} = 2\cdot 91.2\,\text{nm}\ \frac{4}{1} = 2\cdot 365\,\text{nm} = 730\,\text{nm} \quad \text{IR}$$

⇒ Pozitronij ne može emitirati vidljivu svjetlost.

2.

a) $T = 50\,eV$

b) $T = 200\,eV$

λ_{dB}, λ_C = ? (e^-, μ, τ)

r_B = ?
$= a_0$

$$\lambda_{dB} = \frac{h}{p} = \frac{hc}{pc} \qquad \lambda_C = \frac{h}{mc} = \frac{hc}{mc^2}$$

$$hc = 6.626 \cdot 10^{-34}\,J\cancel{s} \cdot 2{,}998 \cdot 10^8\,m/\cancel{s}$$

$$= \frac{6{,}626 \cdot 10^{-34} \cdot 2{,}998 \cdot 10^8}{1.602 \cdot 10^{-19}}\,eVm = 12.40 \cdot 10^{-7}\,eVm$$

$$= 1{,}24 \cdot 10^{-6}\,eVm$$

$$m_e c^2 = 9{,}109 \cdot 10^{-31}\,kg \cdot (2{,}998 \cdot 10^8)^2\,m^2/s^2 = \ldots = 5{,}110 \cdot 10^5\,eV = 511\,keV = \underline{0{,}5\,MeV}$$

$$m_\mu c^2 = 207\,m_e c^2 = 105\,777\,keV = \underline{105{,}8\,MeV} = 1.058 \cdot 10^8\,eV$$

$$m_\tau c^2 = 3477\,m_e c^2 = 1776\,747\,keV = 1776.7\,MeV = 1{,}777 \cdot 10^9\,eV$$
$$= \underline{1{,}777\,GeV}$$

$$\begin{aligned} E &= mc^2 + T \\ E^2 &= m^2c^4 + p^2c^2 \end{aligned} \Rightarrow pc = \sqrt{E^2 - m^2c^4}$$

$$= \sqrt{(mc^2+T)^2 - m^2c^4}$$

$$= \sqrt{T^2 + 2mc^2T}$$

za $T \ll mc^2$

$\approx \sqrt{2mc^2T}$

nerel. slučaj

a) $T = 50\,eV \ll m_e c^2, m_\mu c^2, m_\tau c^2$

b) $T = 200\,eV \ll m_e c^2, m_\mu c^2, m_\tau c^2$

a) $T = 50\,eV$

$$p_e c = \sqrt{2 \cdot 5{,}110 \cdot 10^5 \cdot 50}\ eV = \sqrt{5{,}110 \cdot 10^7}\ eV$$
$$= 7.148 \cdot 10^3 eV$$

$$p_\mu c = \sqrt{207}\ p_e c =$$

$$p_\sigma c = \sqrt{3477}\ p_e c =$$

$$\lambda_{dB}^e = \frac{hc}{p_e c} = \frac{1.24 \cdot 10^{-6} eV m}{7.148 \cdot 10^3 eV} = 1{,}735 \cdot 10^{-10} m$$

$$\lambda_{dB}^\mu = \frac{1}{\sqrt{207}}\ \lambda_{dB}^e = 1{,}206 \cdot 10^{-11} m$$

$$\lambda_{dB}^\sigma = \frac{1}{\sqrt{3477}}\ \lambda_{dB}^e = 2{,}942 \cdot 10^{-12} m$$

b) $T = 200\ eV = 4 \cdot 50 eV$

$$p_e c^{(200)} = \sqrt{4}\ p_e c^{50} = 2 p_e c^{(50)}$$

$$p_\mu c^{(200)} = 2 \cdot p_\mu c^{(50)}$$

$$p_\sigma c^{(200)} = 2 p_\sigma c^{(50)}$$

$$\lambda_{dB}^{e,200} = \frac{1}{2}\ \lambda_{dB}^{e,50} = 8.675 \cdot 10^{-11} m$$

$$\lambda_{dB}^{\mu,200} = \frac{1}{2}\ \lambda_{dB}^{\mu,50} = 6.03 \cdot 10^{-12} m$$

$$\lambda_{dB}^{\sigma,200} = \frac{1}{2}\ \lambda_{dB}^{\sigma,50} = 1.47 \cdot 10^{-12} m$$

$$\lambda_c = \frac{hc}{mc^2}$$

$$\lambda_c^{(e)} = \frac{1{,}240 \cdot 10^{-6} eV m}{5{,}110 \cdot 10^5 eV} = 0{,}2427 \cdot 10^{-11} m = 2{,}427 \cdot 10^{-12} m$$

$$\lambda_c^{(\mu)} = \frac{1}{207}\ \lambda_c^{(e)} = 1{,}172 \cdot 10^{-14} m$$

$$\lambda_c^{(\sigma)} = \frac{1}{3477}\ \lambda_c^{(e)} = 6{,}98 \cdot 10^{-16} m$$

2a

$$a_0^H = \frac{\hbar}{m_e c \alpha} = 0.529 \cdot 10^{-10} m$$

$$\Uparrow$$

$$\mu = \frac{m_e m_p}{m_e + m_p} = \frac{m_e}{1 + \frac{m_e}{m_p}} \approx m_e$$

reducirana masa

jer $m_p >> m_e$

$$\Downarrow$$

$$\mu^{e^+e^-} = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2} \Rightarrow a_0^{e^+e^-} = a_0^H \frac{m_e}{m_e/2} = 2 a_0^H$$

$$= 1,058 \cdot 10^{-10} m$$

$$\mu^{\mu^+\mu^-} = \frac{m_{\mu^+} m_{\mu^-}}{m_{\mu^+} + m_{\mu^-}} = \frac{m_\mu}{2} = \frac{207 m_e}{2} \Rightarrow a_0^{\mu^+\mu^-} = \frac{2}{207} a_0^H = \frac{1}{207} a_0^{e^+e^-}$$

$$= 5,1 \cdot 10^{-13} m$$

$$\mu^{\tau^+\tau^-} = \frac{m_{\tau^+} m_{\tau^-}}{m_{\tau^+} + m_{\tau^-}} = \frac{m_\tau}{2} = 3477 \frac{m_e}{2}$$

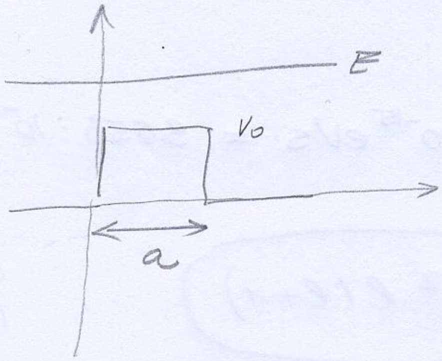
$$\Rightarrow a_0^{\tau^+\tau^-} = \frac{1}{3477} a_0^{e^+e^-}$$

$$= 3.04 \cdot 10^{-14} m$$

3.

$E = \frac{3V_0}{2} \quad > V_0$

$T = 1 \rightarrow a_{min} = ?$



$E > V_0$

$$\left[\ \frac{1}{T} = 1 + \frac{1}{4}\,\frac{V_0^2}{E(E-V_0)}\,\sin^2(\varkappa a)\ \right]$$

$$\varkappa^2 = \frac{2m(E-V_0)}{\hbar^2}$$

$$\cancel{\frac{1}{1}} = \cancel{1} + \frac{1}{4}\,\frac{V_0^2}{E(E-V_0)}\,\sin^2(\varkappa a)$$

$$\Rightarrow \quad \sin^2(\varkappa a) = 0$$

$$\sin(\varkappa a) = 0$$

$$\varkappa a = n\pi$$

$n = \cancel{0}, 1, 2, \ldots$

imamo barijeru neke širine

$$\varkappa a_{min} = \pi$$

$$a_{min} = \frac{\pi}{\varkappa} = \frac{\pi \hbar}{\sqrt{2m(E-V_0)}} = \frac{\pi\hbar}{\sqrt{2m\,\frac{V_0}{2}}}$$

$$a_{min} = \frac{\pi\hbar}{\sqrt{m V_0}}$$

3a

4.

$L = 2{,}28 \cdot 10^{-15}\, eVs = 3{,}653 \cdot 10^{-34}\, Js$

a) $L^2 = \hbar^2 \, l(l+1)$ $\quad \{ \hbar = 1{,}055 \cdot 10^{-34}\, Js$

$l(l+1) = \frac{L^2}{\hbar^2} = \frac{3{,}653^2}{1{,}055^2} \approx 12$

$\underline{\underline{l = 3}}$ $\rightarrow l \in \{0, 1, \ldots, n-1\}$

$m \in \{-l, \ldots, l\} \Rightarrow \underline{\underline{m \in \{-3, -2, -1, 0, 1, 2, 3\}}}$

b) $I = 2{,}66 \cdot 10^{-47}\, kg\,m^2$ $\quad \{ I = m r^2$ (moment inercije

$E_{lm} = E_l = \frac{L^2}{2I} = \frac{\hbar^2 l(l+1)}{2I}$

$E_3 = \frac{(1{,}055 \cdot 10^{-34}\, Js)^2}{2 \cdot 2{,}66 \cdot 10^{-47}\, kg\,m^2} \cdot 12 = 0{,}2092 \cdot 10^{-21}\, J \cdot 12$

$= \underline{\underline{2{,}5106 \cdot 10^{-21}\, J}}$

$\{ \frac{(Js)^2}{kg\,m^2} = \frac{\left(kg \frac{m^2}{s^2} \cancel{s}\right)^2}{kg\,m^2} = kg \frac{m^2}{s^2} = J$

c) $l=3$

$m \in \{-3, -2, -1, 0, 1, 2, 3\}$

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$$

$$Y_3^3(\theta,\varphi) = \sqrt{\frac{7}{4\pi}\frac{0!}{6!}}\left(-15\,(1-\cos^2\theta)\right)^{3/2}\Big) e^{i3\varphi}$$

$$= -\sqrt{\frac{7}{4\pi}\frac{1}{\underbrace{6!}_{=2\cdot3\cdot4\cdot5\cdot6}}}\cdot \overset{3\cdot5}{15}\cdot(\underbrace{1-\cos^2\theta}_{\sin^2\theta})^{3/2}\, e^{i3\varphi} = \frac{-1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{i3\varphi}$$

$$Y_3^{-3}(\theta,\varphi) = (-1)^{-3}\sqrt{\frac{7}{4\pi}\frac{6!}{0!}}\,\frac{15}{720}\,(1-\cos^2\theta)^{3/2}\, e^{-i3\varphi}$$

$$= \sqrt{\frac{7}{4\pi}\frac{6!}{1}}\,\frac{+15}{\underbrace{720}_{6!}}\,(\underbrace{1-\cos^2\theta}_{\sin^2\theta})^{3/2} e^{-i3\varphi}$$

$$= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{-i3\varphi}$$

$$\varphi(\theta,\varphi) = A\left(Y_3^3(\theta,\varphi) + Y_3^{-3}(\theta,\varphi)\right)$$

$$\langle\varphi|\varphi\rangle = 1$$

$$A^2\langle Y_3^3(\theta,\varphi) + Y_3^{-3}(\theta,\varphi) | Y_3^3(\theta,\varphi) + Y_3^{-3}(\theta,\varphi)\rangle = 1$$

$$A^2\Big(\overset{=1}{\langle Y_3^3|Y_3^3\rangle} + \overset{=1\ \text{(kugl. funk. normirane na 1)}}{\langle Y_3^{-3}|Y_3^{-3}\rangle} + \overset{=0}{\langle Y_3^3|Y_3^{-3}\rangle} + \overset{=0}{\langle Y_3^{-3}|Y_3^3\rangle}\Big) = 1$$

$$A^2\cdot 2 = 1 \implies A = \frac{1}{\sqrt{2}}$$