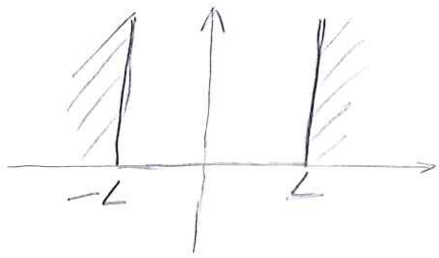
(*) 1.

$$\psi(x,0) = \sqrt{\frac{1}{45L}}\left[6\sin\frac{\pi x}{L} + 3\cos\frac{\pi x}{2L}\right]$$

$$V(x) = \begin{cases} 0 & -L < x < L \\ \infty & \text{inače} \end{cases}$$



$$\begin{cases} \varphi_n = \sqrt{\frac{1}{L}}\cos\frac{n\pi}{2L}x & n = 1, 3, 5, \dots \\ \tilde{\varphi}_n = \sqrt{\frac{1}{L}}\sin\frac{n\pi}{2L}x & n = 2, 4, 6, \dots \end{cases}$$

$$\left\{ E_n = \frac{\hbar^2\pi^2}{2m(2L)^2}n^2 = \frac{\hbar^2\pi^2}{8mL^2}n^2 \right.$$

a)

$$\psi(x,0) = \sqrt{\frac{1}{45L}}\left[6\sqrt{L}\,\tilde{\varphi}_2(x) + 3\sqrt{L}\,\varphi_1(x)\right]$$

$$= \frac{1}{\sqrt{45}}\left[6\tilde{\varphi}_2(x) + 3\varphi_1(x)\right]$$

$$\Rightarrow \psi(x,t) = \frac{1}{\sqrt{45}}\left[6\tilde{\varphi}_2(x)e^{-i\frac{E_2}{\hbar}t} + 3\varphi_1 e^{-i\frac{E_1}{\hbar}t}\right]$$

$$E_1 = \frac{\hbar^2\pi^2}{8mL^2}, \quad E_2 = \frac{\hbar^2\pi^2}{8mL^2}4 = \frac{\hbar^2\pi^2}{2mL^2}$$

b)

$$P(E_1) = \left(\frac{3}{\sqrt{45}}\right)^2 = \underline{\underline{\frac{9}{45}}}\ \frac{1}{5} \leftarrow E_1 = \frac{\hbar^2\pi^2}{8mL^2}$$

$$P(E_2) = \left(\frac{6}{\sqrt{45}}\right)^2 = \underline{\underline{\frac{36}{45}}}\ \frac{4}{5} \leftarrow E_2 = \frac{\hbar^2\pi^2}{2mL^2}$$

$\underline{d)}$ izmjereno u $t=0$ $E=E_2$

$$\Rightarrow \psi(x, t>0) = \tilde{\varphi}_2(x)\, e^{-i\frac{E_2}{\hbar}t}$$

$\underline{c)}$ $\mathbb{P}$... operator pariteta

$$\mathbb{P}\varphi_n = \varphi_n$$

$$\mathbb{P}\tilde{\varphi}_n = -\tilde{\varphi}_n$$

$$\langle \mathbb{P} \rangle = \int \psi^*(x,t)\, \mathbb{P}\, \psi(x,t)\, dx = \langle \psi(x,t) | \mathbb{P} | \psi(x,t) \rangle$$

$$= \frac{1}{45} \langle 6\tilde{\varphi}_2 + 3\varphi_1 | \mathbb{P} | 6\tilde{\varphi}_2 + 3\varphi_1 \rangle =$$

$$= \Big/ \langle \varphi_n | \varphi_m \rangle = \delta_{nm} \quad \text{ortonormirana stanja} \Big/$$

$$= \frac{1}{45} \Big\{ 36 \overbrace{\langle \tilde{\varphi}_2 | \underbrace{\mathbb{P} | \tilde{\varphi}_2 \rangle}_{-|\tilde{\varphi}_2\rangle}}^{-1} + 9 \overbrace{\langle \varphi_1 | \underbrace{\mathbb{P} | \varphi_1 \rangle}_{|\varphi_1\rangle}}^{1} +$$

$$+ 18 \Big[\underbrace{\langle \tilde{\varphi}_2 | \mathbb{P} | \varphi_1 \rangle}_{=0} + \underbrace{\langle \varphi_1 | \mathbb{P} | \tilde{\varphi}_2 \rangle}_{=0} \Big] \Big\}$$

$$= \frac{1}{45}(-36+9) = \underline{\underline{-\frac{27}{45}}} = \underline{\underline{-\frac{3}{5}}}$$

e)

$$\int_{-L/2}^{L/2} \psi^*(x,t>0)\, \psi(x,t>0)\, dx =$$

$$= \int_{-L/2}^{L/2} |\tilde{\varphi}_2|^2 \, dx =$$

$$= \frac{1}{L} \int_{-L/2}^{L/2} \sin^2\left(\frac{\cancel{2}\pi}{\cancel{2}L} x\right) dx = \left| \begin{array}{l} \cos 2t = \cos^2 t - \sin^2 t \\ \quad = 1 - 2\sin^2 t \\ \Rightarrow \sin^2 t = \frac{1-\cos 2t}{2} \end{array} \right|$$

$$= \frac{1}{2L} \left[\int_{-L/2}^{L/2} \left(1 - \cos \frac{2\pi}{L} x\right) dx \right]$$

$$= \frac{1}{2L} \left[\int_{-L/2}^{L/2} dx - \int_{-L/2}^{L/2} dx \, \cos\left(\frac{2\pi}{L} x\right) \right] =$$

$$= \frac{1}{2L} \left[x - \frac{L}{2\pi} \sin \frac{2\pi}{L} x \right]_{-L/2}^{L/2} =$$

$$= \frac{1}{2L} \left[\frac{L}{2} - \left(-\frac{L}{2}\right) - \frac{L}{2\pi} \underbrace{\left(\sin \pi - \sin(-\pi) \right)}_{=0} \right]$$

$$= \frac{1}{2L} \cdot L = \frac{1}{2}$$

(*) 2.

$m = 0.1g = 10^{-4} kg$

$v = 320 \, m/s$

$V_0 = 6 J$

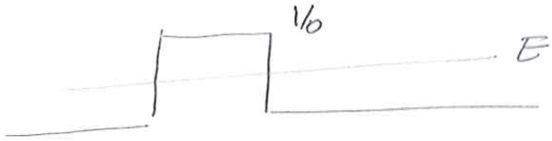
$T = \frac{1}{1000} = 10^{-3}$

$L = ?$

$$E = \frac{mv^2}{2} = \frac{10^{-4}kg \, (320 \, m/s)^2}{2} = 5,12 J$$

$\Downarrow$

$E < V_0$



$$\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(V_0 - E)} sh^2(\varkappa L)$$

$$\varkappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$\varkappa = \sqrt{\frac{2 \cdot 10^{-4}kg \, (6 - 5,12) J}{1.054 \cdot 10^{-34} Js}}$$

$$= 1,26 \cdot 10^{32} \sqrt{\frac{kg \, J}{Js}} \; m^{-1}$$

$$\sqrt{\frac{kg \, \cancel{kg} \frac{m^2}{\cancel{s^2}}}{\cancel{kg} \frac{m^2}{\cancel{s^2}} s}} = \frac{1}{m}$$

$$sh^2 \varkappa L = \frac{4E(V_0 - E)}{V_0^2} \left(\frac{1}{T} - 1 \right)$$

$$sh \, \varkappa L = \sqrt{\frac{4E(V_0 - E)}{V_0^2} \left(\frac{1}{T} - 1 \right)}$$

$$= \sqrt{\frac{4 \cdot 5,12 (6 - 5,12)}{6^2} (1000 - 1)} = \sqrt{500.122}$$

$$= 22,3634$$

$$sh \, x = \frac{e^x - e^{-x}}{2}$$

$sh \, x = A$

$e^x - e^{-x} = 2A \qquad e^x = t$

$t - 1/t = 2A$

$\Rightarrow t^2 - 2At - 1 = 0$

$$t_{1/2} = \frac{2A \pm \sqrt{4A^2 + 4}}{2} = A \pm \sqrt{A^2 + 1}$$

$t = e^x > 0 \Rightarrow e^x = A + \sqrt{A^2 + 1}$

$\Rightarrow x = \ln(A + \sqrt{A^2 + 1}) = arcsh \, A$

$\varkappa L = arcsh \, 22.3634$

$\varkappa L = 3.80107$

$= \ln(22,3634 + \sqrt{22,3634^2 + 1})$

$= \ln 44,7491$

$$L = \frac{1}{\varkappa} 3,80107$$

$$= 3,02 \cdot 10^{-32} m$$

(*) 3.

$$\psi(x,0) = \sqrt{2}\,A\,\phi_1(x) + \frac{1}{\sqrt{2}}\,A\,\phi_2(x) + A\,\phi_3(x)$$

$$\begin{cases} \phi_n(x) = A_n H_n(\xi)\, e^{-\xi^2/2}, \quad \xi = \sqrt{\frac{m\omega_0}{\hbar}}\, x_0 \\ \qquad A_n = \sqrt{\sqrt{\frac{m\omega_0}{\hbar\pi}}\,\frac{1}{2^n n!}} \qquad n = 0,1,2,\dots \\ E_n = \hbar\omega_0 (n + 1/2) \end{cases}$$

$$\left\{ \int_{-\infty}^{\infty} \phi_n(x)\,\phi_m(x)\,dx = \delta_{nm} \right. \quad \text{ortonormirana stanja}$$

$$\left\{ \Phi_n(x,t) = \phi_n(x)\, e^{-i E_n/\hbar\, t} \right.$$

a) $$\int_{-\infty}^{\infty} \psi^*(x,0)\,\psi(x,0)\,dx = 1$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} \left[\sqrt{2}\,A\,\phi_1(x) + \frac{1}{\sqrt{2}}\,A\,\phi_2(x) + A\,\phi_3(x)\right]^2 dx = 1$$

$$\Downarrow \quad \int \phi_n(x)\,\phi_m(x)\,dx = \langle \phi_n | \phi_m \rangle = \delta_{nm}$$

$$2A^2 + \frac{1}{2}A^2 + A^2 = 1$$

$$A^2\left(3 + \frac{1}{2}\right) = 1 \implies A = \sqrt{\frac{2}{7}}$$

$$\psi(x,0) = \frac{2}{\sqrt{7}}\,\phi_1(x) + \frac{1}{\sqrt{7}}\,\phi_2(x) + \sqrt{\frac{2}{7}}\,\phi_3(x)$$

b) $$\psi(x,t) = \frac{2}{\sqrt{7}}\,\phi_1(x)\,e^{-i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{7}}\,\phi_2(x)\,e^{-i\frac{E_2}{\hbar}t} + \sqrt{\frac{2}{7}}\,\phi_3(x)\,e^{-i\frac{E_3}{\hbar}t}$$

$$= \left| E_1 = \hbar\omega_0\left(1 + \frac{1}{2}\right) = \frac{3}{2}\hbar\omega_0, \; E_2 = \hbar\omega_0\frac{5}{2}, \; E_3 = \hbar\omega_0\frac{7}{2} \right|$$

$$= \frac{2}{\sqrt{7}}\,\phi_1(x)\,e^{-i\frac{3}{2}\omega_0 t} + \frac{1}{\sqrt{7}}\,\phi_2(x)\,e^{-i\frac{5}{2}\omega_0 t} + \sqrt{\frac{2}{7}}\,\phi_3(x)\,e^{-i\frac{7}{2}\omega_0 t}$$

c)

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx =$$

$$= \int_{-\infty}^{\infty} dx\, x \left| \frac{2}{\sqrt{7}}\phi_1(x) e^{-i\frac{3}{2}\omega_0 t} + \frac{1}{\sqrt{7}}\phi_2(x) e^{-i\frac{5}{2}\omega_0 t} + \right.$$

$$\left. + \sqrt{\frac{2}{7}}\phi_3(x) e^{-7/2\,\omega_0 t}\right|^2 =$$

$$= \left| x = \sqrt{\frac{\hbar}{m\omega_0}}\xi \ , \ dx = \sqrt{\frac{\hbar}{m\omega_0}} d\xi \right| =$$

$$= \frac{\hbar}{m\omega_0}\int_{-\infty}^{\infty} d\xi\ \xi \left| \frac{2}{\sqrt{7}} A_1 H_1(\xi) e^{-\xi^2/2} e^{-i\,3/2\,\omega_0 t} + \right.$$

$$+ \frac{1}{\sqrt{7}} A_2 H_2(\xi) e^{-\xi^2/2} e^{-i\,5/2\,\omega_0 t} +$$

$$\left. + \sqrt{\frac{2}{7}} A_3 H_3(\xi) e^{-\xi^2/2} e^{-i\,7/2\,\omega_0 t}\right|^2 =$$

$$= \left| \int_{-\infty}^{\infty} d\xi\, H_n(\xi) H_m(\xi) e^{-\xi^2} = \delta_{nm}\sqrt{\pi}\, 2^n n! \quad (*) \right.$$

$\Rightarrow$ $\xi H_n(\xi)$ faktor transformiramo koristeći rekurzivnu relaciju

$$\xi H_n(\xi) = \frac{1}{2} H_{n+1}(\xi) + n H_{n-1}(\xi)$$

$\Rightarrow$ uzevši u obzir $(*)$ vidimo da doprinose samo članovi $\sim \xi H_1 H_2$ i $\xi H_2 H_3$

dok integrali s $\xi H_1 H_1$, $\xi H_2 H_2$, $\xi H_3 H_3$ i $\xi H_1 H_3$ iščezavaju

$$= \frac{\hbar}{m\omega_0}\left\{ \frac{2}{\sqrt{7}}\frac{1}{\sqrt{7}} A_1 A_2 \int_{-\infty}^{\infty} d\xi\, \xi H_1(\xi) H_2(\xi) e^{-\xi^2} \underbrace{\left(e^{-i(\frac{3}{2}-\frac{5}{2})\omega_0 t} + e^{-i(\frac{5}{2}-\frac{3}{2})\omega_0 t}\right)}_{N_{12}(t)} \right.$$

$$+ \frac{1}{\sqrt{7}}\sqrt{\frac{2}{7}} A_2 A_3 \int_{-\infty}^{\infty} d\xi\, \xi H_2(\xi) H_3(\xi) e^{-\xi^2} \underbrace{\left(e^{-i(\frac{5}{2}-\frac{7}{2})\omega_0 t} + e^{-i(\frac{7}{2}-\frac{5}{2})\omega_0 t}\right)}_{N_{23}(t)}$$

$$\left. + 0 \right\} = \left| \begin{array}{l} N_{12}(t) = e^{-i\omega_0 t} + e^{i\omega_0 t} = 2\cos\omega_0 t \\ N_{23}(t) = e^{-\omega_0 t} + e^{i\omega_0 t} = 2\cos\omega_0 t \end{array} \right|$$

$$= \frac{\hbar}{m\omega_0} \left\{ \frac{4}{7} A_1 A_2 \; \frac{1}{2} \int_{-\infty}^{\infty} d\xi \, H_2(\xi) H_2(\xi) e^{-\xi^2} + 0 + \right.$$

$$\left. + \frac{2\sqrt{2}}{7} A_2 A_3 \; \frac{1}{2} \int_{-\infty}^{\infty} d\xi \, H_3(\xi) H_3(\xi) e^{-\xi^2} + 0 \right\} \cos\omega_0 t$$

$$= \frac{\hbar}{m\omega_0} \left[\frac{2}{7} A_1 A_2 \sqrt{\pi} \, 2^2 2! + \right.$$

$$\left. + \frac{\sqrt{2}}{7} A_2 A_3 \sqrt{\pi} \, 2^3 3! \right] \cos\omega_0 t$$

$$= \frac{\hbar}{m\omega_0} \sqrt{\frac{m\omega_0}{\hbar \cancel{\pi}}} \cancel{\sqrt{\pi}} \left[\frac{2}{7} \sqrt{\frac{1}{2^1 1!}} \sqrt{\frac{1}{2^2 2!}} \, 2^2 2! + \right.$$

$$\left. + \frac{\sqrt{2}}{7} \sqrt{\frac{1}{2^2 2!}} \sqrt{\frac{1}{2^3 3!}} \, 2^3 3! \right] \cos\omega_0 t$$

$$= \sqrt{\frac{\hbar}{m\omega_0}} \left[\frac{2^4}{7 \cdot 2^2} + \frac{\cancel{\sqrt{2}}}{7} \frac{2^4 \cdot 3}{2^3 \cancel{\sqrt{2}} \cdot \sqrt{3}} \right] \cos\omega_0 t$$

$$= \sqrt{\frac{\hbar}{m\omega_0}} \left(\frac{4}{7} + \frac{2\sqrt{3}}{7} \right) \cos\omega_0 t$$

$$= \sqrt{\frac{\hbar}{m\omega_0}} \, \frac{2}{7} \left(2 + \sqrt{3} \right) \cos\omega_0 t$$

$$\left\{ \sqrt{\frac{Js}{kg\, s^{-1}}} = \sqrt{\frac{\cancel{kg}\, \frac{m^2}{s^2}\, s}{\cancel{kg}\, s^{-1}}} = m \right\}$$

(*) 4.

$$\psi(\theta,\varphi) = \frac{1}{\sqrt{7}} Y_{2,-1}(\theta,\varphi) + \sqrt{\frac{4}{5}}\, Y_{2,0}(\theta,\varphi) + \sqrt{\frac{2}{35}}\, Y_{2,2}(\theta,\varphi)$$

$$\left\{ \underbrace{\int d\Omega\, Y_{lm} Y_{l'm'}}_{\langle lm | l'm' \rangle} = \delta_{ll'} \delta_{mm'} \right. \longrightarrow |l,m\rangle = Y_l^m(\theta,\varphi)$$

kugline funkcije

$$\left\{ \begin{aligned} L^2 |l,m\rangle &= \hbar^2 l(l+1) |l,m\rangle \\ L_z |l,m\rangle &= \hbar m |l,m\rangle \end{aligned} \right.$$

$$\left\{ E = \frac{L^2}{2I} = \frac{\hbar^2 l(l+1)}{2mr^2} \right.$$

a) $l = 2$

$m = -1, 0, 2$

$L_z = \hbar(-1)$ $\qquad P(L_z = -\hbar) = \left(\frac{1}{\sqrt{7}}\right)^2 = \frac{1}{7}$

$L_z = 0$ $\qquad P(L_z = 0) = \frac{4}{5}$

$L_z = 2\hbar$ $\qquad P(L_z = 2\hbar) = \frac{2}{35}$

b)

$$\cos\alpha = \frac{L_z}{|\vec{L}|} = \frac{m\cancel{\hbar}}{\sqrt{l(l+1)}\,\cancel{\hbar}}$$

$$(L_z)_{max} = 2\hbar$$

$$\cos\alpha = \frac{2\hbar}{\sqrt{2(2+1)}\,\hbar} = \frac{2}{\sqrt{6}} = 0{,}816497$$

$$\alpha = \arccos\frac{2}{\sqrt{6}} = 0{,}61548 = 35.26^\circ$$

c) $E = \frac{\hbar^2 l(l+1)}{2I}$ $I = mr^2$ moment inercije

$l = 2 \Rightarrow \psi(\theta,\varphi)$ je svojstveno stanje energije

d)

$$\langle L^2\rangle = \langle \frac{1}{\sqrt{7}} Y_{2,-1} + \sqrt{\frac{4}{5}} Y_{2,0} + \sqrt{\frac{2}{35}} Y_{2,2}(\theta,\varphi) | L^2 | \frac{1}{\sqrt{7}} Y_{2,-1} + \sqrt{\frac{4}{5}} Y_{2,0} + \sqrt{\frac{2}{35}} Y_{2,2}\rangle$$

$$= \hbar^2 2(2+1) \left\{ \frac{1}{7}\langle 2,-1|2,-1\rangle + \frac{4}{5}\langle 2,0|2,0\rangle + \frac{2}{35}\langle 2,2|2,2\rangle + 0 \right\}$$

$$= \underline{\underline{6\hbar^2}}$$

$$\langle L_z\rangle = \langle \frac{1}{\sqrt{7}} Y_{2,-1} + \sqrt{\frac{4}{5}} Y_{2,0} + \sqrt{\frac{2}{35}} Y_{2,2}(\theta,\varphi) | L_z | \frac{1}{\sqrt{7}} Y_{2,-1} + \sqrt{\frac{4}{5}} Y_{2,0} + \sqrt{\frac{2}{35}} Y_{2,2}\rangle$$

$$= \hbar\left[\frac{1}{7}(-1)\langle 2,-1|2,-1\rangle + \frac{4}{5}\cdot 0\,\langle 2,0|2,0\rangle + \frac{2}{35}\cdot 2\,\langle 2,2|2,2\rangle\right] =$$

$$= \hbar\left(-\frac{1}{7} + \frac{4}{35}\right) = \underline{\underline{-\frac{1}{35}\hbar}}$$