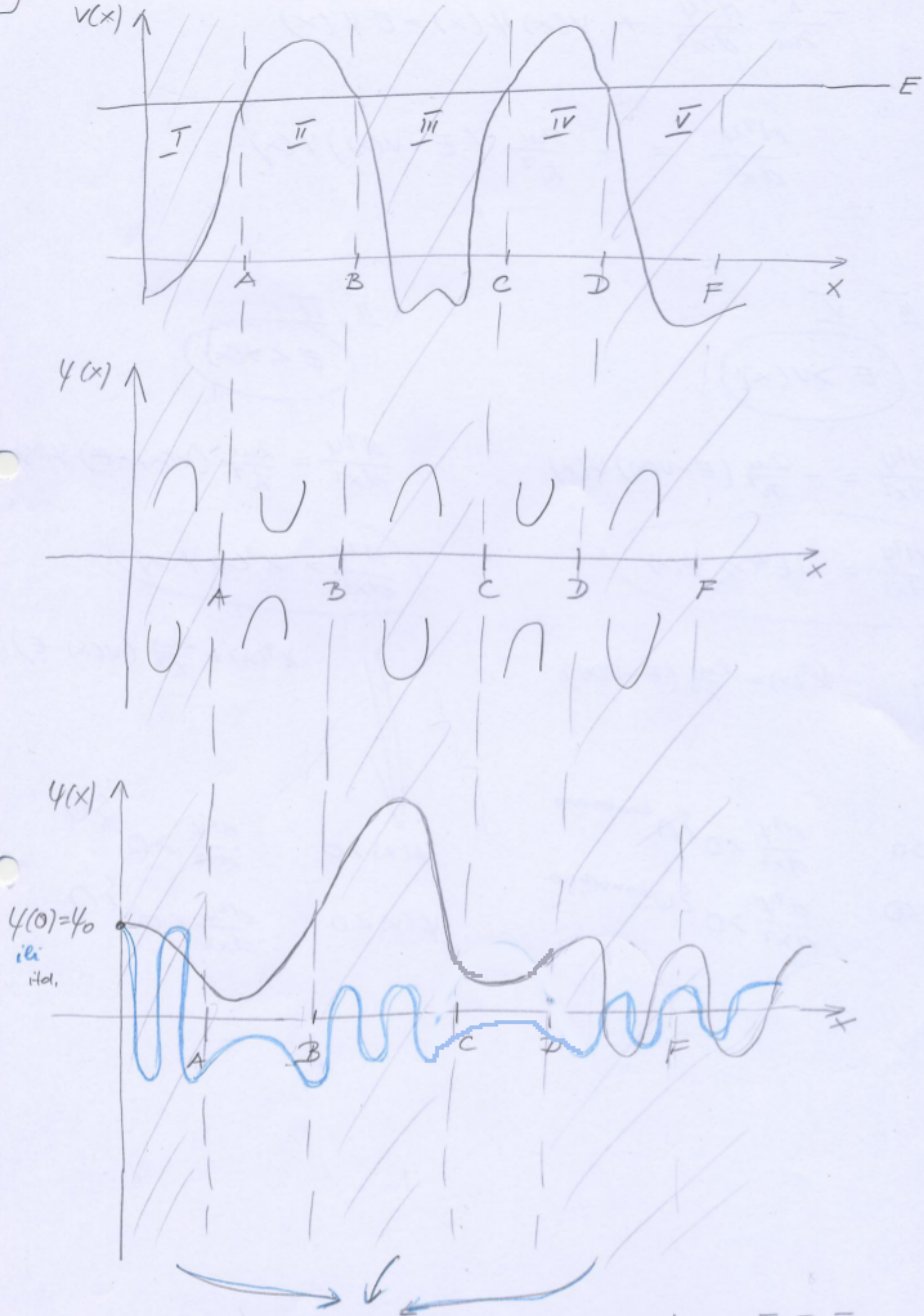
1.



moguće oscilatorno ponašanje u I, III, V

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x)$$

$\overline{I}, \overline{III}, \overline{V}$

$E > V(x)$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -k^2(x)\,\psi(x)$$

$$k^2(x) = \frac{2m}{\hbar^2}\left(E - V(x)\right)$$

$\psi(x) > 0 \qquad \frac{d^2\psi}{dx^2} < 0$ { $\cap$ koukarro

$\psi(x) < 0 \qquad \frac{d^2\psi}{dx^2} > 0$ { $\cup$ konvekso

$\overline{II}, \overline{IV}$

$E < V(x)$

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\left(V(x) - E\right)\psi(x)$$

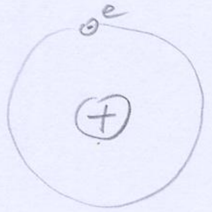
$$\frac{d^2\psi}{dx^2} = \varkappa^2(x)\,\psi(x)$$

$$\varkappa^2(x) = \frac{2m}{\hbar^2}\left(V(x) - E\right)$$

$\psi(x) > 0 \qquad \frac{d^2\psi}{dx^2} > 0$ { $\cup$

$\psi(x) < 0 \qquad \frac{d^2\psi}{dx^2} < 0$ { $\cap$

$\boxed{2.}$



Ako elektron kruži oko protona, tada je akceleriran pri svakoj promjeni smjera te zrači.
Klasičan račun daje rezultat prema kojem će elektron izgubiti postepeno svoju rotacijsku kin. energiju te pasti na proton (za 10^{-9} s).

To se ne događa. Vodikov atom je stabilan.
Pokušaj objašnjenja koristeći H. relacije neodređenosti:

$$E = \frac{p^2}{2\mu} - \frac{e^2}{k_c r}$$

klasično: minimum energije vodikovog atoma je $-\infty$ za $p=0$, $r=0$ (e^- na p)

kv. mehanika: princip neodređenosti tjera da elektron ima impuls i poziciju različitu od 0

Neka je R prosječna udaljenost e^- i p.

H. princip neodređenosti $\quad \Delta p \cdot \Delta r \geq \frac{\hbar}{2}$

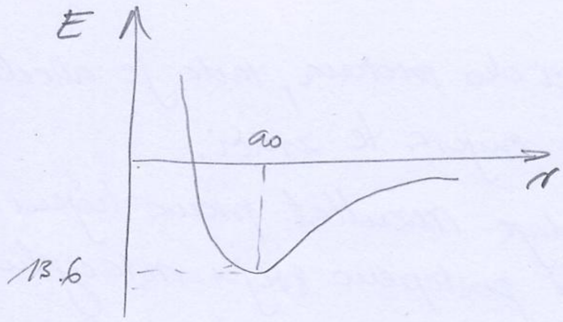
$$\underbrace{\Delta p}_{\approx p} \cdot R = \hbar/2 \quad \Rightarrow p = \frac{\hbar}{2R}$$

$$\Rightarrow E = \frac{\hbar^2}{8R^2\mu} - \frac{e^2}{k_c R} \quad \Rightarrow \quad E_{R\to 0} = +\infty$$ nije energ. povoljno

$$\Rightarrow \frac{dE}{dR} = -2\frac{\hbar^2}{8R^3\mu} + \frac{e^2}{k_c R^2} \overset{!}{=} 0 \quad / 8k_c\mu R^3 \quad \Rightarrow -2\hbar^2 k_c + 8\mu R e^2 = 0$$

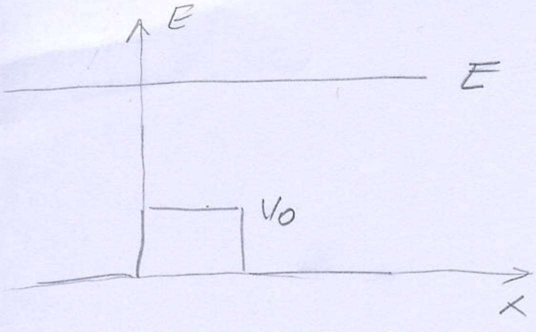
$$\Rightarrow R = \frac{\hbar^2 k_c}{4\mu e^2} \sim a_0$$

minimum i stabilnost



$\Rightarrow$ kvantna mehanika kaže da vodikov atom ne može kolapsirati

1.



$$\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2(\varkappa L)$$

$$\varkappa^2 = \frac{2m(E-V_0)}{\hbar^2}$$

$T = 1$

$\Rightarrow \quad \sin \varkappa L = 0$

$\lim\limits_{x\to 0} \frac{\sin^2 x}{x^2} = 1$ ✓

$x \sim \sqrt{E-V_0}$

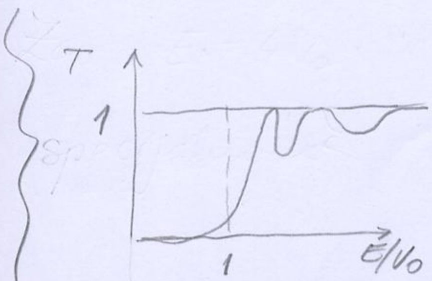
$\Rightarrow \varkappa L = n\pi \qquad n = \cancel{0}, 1, 2, \dots$ → ostalom $E > V_0$

$$\sqrt{\frac{2m}{\hbar^2}(E_{T=1}-V_0)}\, L = n\pi \quad \Rightarrow \quad \frac{2m}{\hbar^2}(E-V_0) = \frac{n^2\pi^2}{L^2}$$

$$\Rightarrow \quad E_{T=1} = n^2 \frac{\hbar^2\pi^2}{2mL^2} + V_0 \,, \quad n = 1, 2, \dots$$

tj. $T = 1$ za $E_{T=1} = V_0 + \frac{\hbar^2\pi^2}{2mL^2}$,

$E_{T=1} = V_0 + 4\frac{\hbar^2\pi^2}{2mL^2}, \dots$



(2.)

a)

$$\int_0^L \sin\frac{n\pi x}{L}\, x(x-L)\, dx =$$

BRONŠTAJN, str. 427–428 (279), 280))

$$= \left| \begin{aligned} &\int x \sin ax\, dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} \\ &\int x^2 \sin ax\, dx = \frac{2x}{a^2}\sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos ax \end{aligned} \right|$$

$$\int_0^L dx\, x^2 \sin\left(\frac{n\pi}{L}x\right) =$$

$$= \frac{2x}{\left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^L - \left(\frac{x^2}{\frac{n\pi}{L}} - \frac{2}{\left(\frac{n\pi}{L}\right)^3}\right)\cos\frac{n\pi}{L}x\Big|_0^L$$

$$= 0 - 0 - \left(\frac{L^2}{n\pi/L} - \frac{2}{\left(\frac{n\pi}{L}\right)^3}\right)\cos n\pi - \frac{2}{\left(\frac{n\pi}{L}\right)^3}\cos 0$$

$$= \left(-\frac{L^3}{n\pi} + \frac{2L^3}{(n\pi)^3}\right)\cos n\pi - \frac{2L^3}{(n\pi)^3}$$

$$= \begin{cases} -\dfrac{L^3}{n\pi} & n \text{ paran} \\ \dfrac{L^3}{n\pi} - \dfrac{4L^3}{(n\pi)^3} & n \text{ neparan} \end{cases}$$

$$\int_0^L dx\, x \sin\left(\frac{n\pi}{L}x\right) = \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\Big|_0^L - \frac{x\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\Big|_0^L$$

$$= 0 - \frac{L^2\cos n\pi}{n\pi} + 0 = \begin{cases} -L^2/n\pi & n \text{ paran} \\ L^2/n\pi & n \text{ neparan} \end{cases}$$

$$\Downarrow$$

$$\int_0^L \sin\frac{n\pi}{L}\, x(x-L) =$$

$$= \begin{cases} -\frac{L^3}{n\pi} - L\left(-\frac{L^2}{n\pi}\right) = 0 & n \text{ paran} \\ \cancel{\frac{L^3}{n\pi}} - \frac{4L^3}{(n\pi)^3} - \cancel{L\cdot\frac{L^2}{n\pi}} = -\frac{4L^3}{(n\pi)^3} & n \text{ neparan} \end{cases}$$

$$\Downarrow$$

$$\psi(x,0) = A\, x(x-L) \qquad A = \sqrt{\frac{30}{L^5}}$$

$$= \sum_n b_n(0)\, \psi_n(x)$$

$$\begin{cases} \psi_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}, & n = 1,2,3,\dots \\ E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2 \end{cases}$$

$$b_n(0) = \int_0^L \psi_n^*(x)\, \psi(x,t)\, dx =$$

$$= \sqrt{\frac{2}{L}} A \int_0^L dx\, \sin\frac{n\pi x}{L}\, x(x-L) =$$

$$= \sqrt{\frac{2}{L}} A \begin{cases} 0 & n \text{ paran} \\ -\frac{4L^3}{(n\pi)^3} & n \text{ neparan} \end{cases}$$

$$\Downarrow$$

$$\left[\begin{aligned} \psi(x,0) &= A\, x(x-L) \\ &= \underbrace{A\sqrt{\frac{2}{L}}}_{\sqrt{\frac{30}{L^5}}\cdot\sqrt{\frac{2}{L}} = \sqrt{\frac{60}{L^6}}} \sum_{n=1,3,..} \left(-\frac{4L^3}{(n\pi)^3}\right)\psi_n(x) \end{aligned}\right]$$

b)

$$P(E_n) = |b_n(0)|^2 =$$

$$= A^2\left(\sqrt{\frac{2}{L}}\right)^2 \left(-\frac{4L^3}{(n\pi)^3}\right)^2 =$$

$$= \frac{60}{\cancel{L^6}}\,\frac{16\cancel{L^6}}{n^6\pi^6} = \overset{0.99856}{\frac{960}{\pi^6}}\,\frac{1}{n^6} \qquad n \text{ neparan}$$

$P(E_n) = 0$ $\quad$ n paran

$N(E_n) = P(E_n)\cdot N_0$

$N_0 = 10^5$

n	$P(E_n)$	$N(E_n)$
1	0.99856	99856
2	0	0
3	0.00137	137
4	0	0
5	0,00006	6

c) E_3 izmjereno u $t=0$

$$\Rightarrow \psi(x,t) = \sqrt{\frac{60}{L^6}}\,\psi_3(x)\, e^{-i/\hbar\, E_3 t}$$

$$E_3 = 9\,\frac{\hbar^2\pi^2}{2mL^2}$$

(3.)

1D harmonički oscilator

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\Psi_k(x,t) + \Psi_{k+1}(x,t)\right]$$

$\overline{x(t)} = ?$

a)

$$\Psi_k(x,t) = \psi_k(x)\, e^{-\frac{i}{\hbar}E_k t}$$

$$\Psi_{k+1}(x,t) = \psi_{k+1}(x)\, e^{-\frac{i}{\hbar}E_{k+1} t}$$

$$\left\{ \begin{array}{l} \psi_n(x) = A_n e^{-\xi^2/2} H_n(\xi), \quad \xi = \sqrt{\frac{m\omega_0}{\hbar}}x, \quad \omega_0 = \sqrt{\frac{k}{m}} \\ A_n = \sqrt{\sqrt{\frac{m\omega_0}{\hbar\pi}}\frac{1}{2^n n!}} \\ E_n = \hbar\omega_0(n+1/2), \quad n = 0,1,2\ldots \end{array} \right.$$

$$\overline{x(t)} = \int_{-\infty}^{\infty} dx\, \Psi^*(x,t)\, x\, \Psi(x,t) =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx\left[\Psi_k^*(x,t) + \Psi_{k+1}^*(t)\right] x \left[\Psi_k(x,t) + \Psi_{k+1}(t)\right]$$

$$= \frac{1}{2}\Big[\int_{-\infty}^{\infty} dx\, x\, (\psi_k(x))^2 e^{-\frac{i}{\hbar}\overbrace{(E_k-E_k)}^{=0}t} + \int_{-\infty}^{\infty} dx\, x \cdot (\psi_{k+1}(x))^2 \cdot 1 +$$

$$+ \int_{-\infty}^{\infty} dx\, x\, \psi_k(x)\psi_{k+1}(x)\left(e^{-\frac{i}{\hbar}(E_k-E_{k+1})t} + e^{-\frac{i}{\hbar}(E_{k+1}-E_k)t}\right)$$

$$= \left| \begin{array}{l} E_{k+1} - E_k = \hbar\omega_0(\cancel{k}+1+\cancel{1/2}-\cancel{k}-\cancel{1/2}) = \hbar\omega_0; \\ \xi = \sqrt{\frac{m\omega_0}{\hbar}}x, \quad d\xi = \sqrt{\frac{m\omega_0}{\hbar}}dx \end{array} \right|$$

$$= \frac{1}{2}\left(\sqrt{\frac{\hbar}{m\omega_0}}\right)^2 \Big[A_k^2 \int_{-\infty}^{\infty} d\xi\ \xi\ e^{-\xi^2}\left(H_k(\xi)\right)^2 +$$

$$+ A_{k+1}^2 \int_{-\infty}^{\infty} d\xi\ \xi\ e^{-\xi^2}\left(H_{k+1}(\xi)\right)^2 +$$

$$+ A_k A_{k+1}\left(e^{-i/\hbar(\hbar\omega_0)t} + e^{+i/\hbar(\hbar\omega_0)t}\right)\int_{-\infty}^{\infty} d\xi\, \xi\, e^{-\xi^2} H_k(\xi) \cdot H_{k+1}(\xi)\Big]$$

$$\int_{-\infty}^{\infty} d\xi\ \xi\ e^{-\xi^2}\left(H_k(\xi)\right)^2 = \int_{-\infty}^{\infty} d\xi\ e^{-\xi^2}\left(\xi H_k(\xi)\right) H_k(\xi) =$$

$$= \left| \begin{array}{l} \xi H_k(\xi) = k H_{k-1}(\xi) + \frac{H_{k+1}(\xi)}{2} \\ \int_{-\infty}^{\infty} H_k(\xi) H_{k'}(\xi) e^{-\xi^2} d\xi = \delta_{kk'} \sqrt{\pi}\, 2^n n! \end{array} \right|$$

$*$

$$= 0 + 0 = 0$$

$$\int_{-\infty}^{\infty} d\xi\ \xi\ e^{-\xi^2} H_k(\xi) H_{k+1}(\xi) = \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2}\left(\xi H_k(\xi)\right) H_{k+1}(\xi) =$$

$$= \left| * \right| = 0 + \frac{1}{2}\int_{-\infty}^{\infty} d\xi\ e^{-\xi^2} H_{k+1}(\xi) H_{k+1}(\xi) =$$

$$= \frac{1}{2}\cdot \sqrt{\pi}\, 2^{k+1} (k+1)! = \underline{\underline{\sqrt{\pi}\, 2^k (k+1)!}}$$

$$\Rightarrow \overline{x(t)} = \frac{1}{2}\frac{\hbar}{m\omega_0}\Big[0 + 0 + \overbrace{(e^{-i\omega_0 t} + e^{i\omega_0 t})}^{A_k A_{k+1}} \sqrt{\pi}\, 2^k (k+1)!\Big]$$

$$= \frac{1}{2}\frac{\hbar}{m\omega_0}\left(\sqrt[4]{\frac{m\omega_0}{\hbar\cancel{\pi}}}\right)^2 \sqrt{\frac{1}{\cancel{2^k}\, k!\, \cancel{2^{k+1}} (k+1)!}}\ \cancel{\sqrt{\pi}}\ 2^k (k+1)!\ (e^{-i\omega_0 t} + e^{i\omega_0 t})$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\frac{k+1}{2}}\left(e^{-i\omega_0 t} + e^{i\omega_0 t}\right)$$

$$\Rightarrow \quad \overline{x(t)} = \sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\frac{k+1}{2}} \quad \frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}$$

$$= \sqrt{\frac{\hbar}{m\omega_0}}\sqrt{\frac{k+1}{2}} \cos\omega_0 t$$

b) $\psi(x,t)$ nije stacionarno stanje

#) za stacionarna stanja $\rho(x,t)=\rho(x)$

$$\rho(x,t) = \psi^*(x,t)\,\psi(x,t) =$$

$$= (\psi_k(x))^2 + (\psi_{k+1}(x))^2 + \psi_k(x)\psi_{k+1}(x) \cdot \left(e^{-i/\hbar (E_k - E_{k+1})t} + e^{-i/\hbar (E_{k+1}-E_k)t}\right)$$

$\neq \rho(x)$, tj. ovisi o vremenu

ili

#) $\hat{H}\psi(x,t) = E\psi(x,t)$ za stacionarna stanja

$$\hat{H}\psi(x,t) = \hat{H}\left(\frac{1}{\sqrt{2}}\psi_k(x,t)\right) + \hat{H}\left(\frac{1}{\sqrt{2}}\psi_{k+1}(x,t)\right)$$

$$= \left(E_k\,\psi_k(x,t) + E_{k+1}\,\psi_{k+1}(x,t)\right)\frac{1}{\sqrt{2}}$$

$$\neq \text{cte.}\ \psi(x,t)$$

(4.)

$L = 2.584 \cdot 10^{-34}\,Js$

a) $l, m = ?$

$$L^2 = \hbar^2 l(l+1) \Rightarrow l(l+1) = \frac{L^2}{\hbar^2} = \frac{2.584^2}{1.055^2} \approx 6 = 2(2+1)$$

$$\Downarrow$$

$$\underline{\underline{l = 2}}$$

$$m \in \{-l, \dots l\} = \underline{\underline{\{-2, -1, 0, 1, 0, 2\}}}$$

c) $l = 2$

$m \in \{\underline{\underline{-2}}, -1, 0, 1, \underline{\underline{2}}\}$

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

$$\left\{ \begin{array}{ll} Y_2^2(\theta, \varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\varphi} & Y_2^1(\theta, \varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{i\varphi} \\ Y_2^{-2}(\theta, \varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{-2i\varphi} & Y_2^{-1}(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{-i\varphi} \\ & Y_2^0(\theta, \varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1) \end{array} \right.$$

$$\varphi(\theta, \varphi) = A\left(Y_2^2 + Y_2^1 + Y_2^0 + Y_2^{-1} + Y_2^{-2}\right)(\theta, \varphi)$$

$$\langle \varphi | \varphi \rangle \overset{!}{=} 1$$

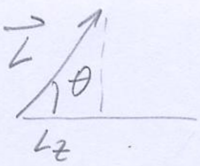
$$A^2 \langle Y_2^2 + Y_2^1 + Y_2^0 + Y_2^{-1} + Y_2^{-2} | Y_2^2 + Y_2^1 + Y_2^0 + Y_2^{-1} + Y_2^{-2} \rangle$$

$= |$ kugline funkcije su ortonormirane $\langle Y_l^m | Y_{l'}^{m'} \rangle = \delta_{ll'}\delta_{mm'} |$

$$= A^2 (1 + 1 + 1 + 1 + 1 + 0 + 0 \dots) \overset{!}{=} 1$$

$$\Rightarrow A = \underline{1/\sqrt{5}}$$

b)



$$\cos\theta = \frac{L_z}{L} = \frac{m\hbar}{\sqrt{l(l+1)}\hbar}$$

$$\theta_{min} \leftarrow \max(\cos\theta) = \max \frac{m}{\sqrt{l(l+1}} = \frac{l}{\sqrt{l(l+1)}} = \frac{2}{\sqrt{6}}$$

$$\Rightarrow \theta_{min} = \arccos\frac{2}{\sqrt{6}} = 0.6155 = \underline{\underline{35,26^\circ}}$$

$$\theta(m,l) = \arccos\frac{m}{\sqrt{l(l+1)}}$$

$l=2$

m	θ
2	$35,26^\circ$
1	$\arccos\frac{1}{\sqrt{6}} = 65.90^\circ$
0	$\arccos 0 = 90^\circ$
-1	$\arccos\frac{-1}{\sqrt{6}} = 114,1$
-2	$\arccos -\frac{2}{\sqrt{6}} = 144,74^\circ$