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(23.6.2015.)

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1.

$E = 1.6 \cdot 10^{-17}\ J$

$V_0 = 90\ eV$

$R = 2/3$

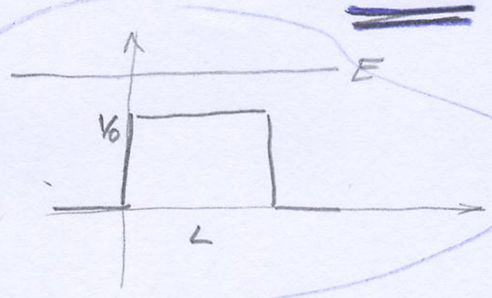
$L = ?$

$e = 1.6 \cdot 10^{-19}\ C$

$E = 1.6 \cdot 10^{-17}\ J \quad J = VC$

$= 1.6 \cdot 10^{-17} \dfrac{e}{1.6 \cdot 10^{-19}} V$

$= \underline{\underline{100\ eV}}$



$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & 0<x<L \\ 0 & x>L \end{cases}$$

$[R = 1-T] \Rightarrow T = 1 - 2/3 = \underline{1/3}$

$$\left[\frac{1}{T} = 1 + \frac{1}{4} \frac{V_0^2}{E(E-V_0)} \sin^2(\kappa L) \qquad \kappa = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right]$$

$\Downarrow$

$$\sin^2(\kappa L) = \left(\frac{1}{T} - 1\right) \cdot \frac{4E(E-V_0)}{V_0^2}$$

$$= (3-1) \frac{4 \cdot \cancel{100}(100-90)}{\underbrace{90^2}_{=81 \cdot \cancel{100}}}$$

$$= \frac{2}{81} \cdot 40 = \frac{80}{81}$$

$$L = \frac{1}{\kappa} \arcsin\sqrt{\frac{80}{81}}$$

$$= 6.18 \cdot 10^{-11}\ m \cdot 1.46$$

$$= \underline{\underline{9.02 \cdot 10^{-11}\ m}}$$

$$\kappa = \sqrt{\frac{2 \cdot 9.1 \cdot 10^{-31} kg\ 10 \cdot 1.6 \cdot 10^{-19} J}{(1.054 \cdot 10^{-34}\ Js)^2}}$$

$$= \left| \frac{kg\ \cancel{J}}{J^{\cancel{2}} s^2} = \frac{\cancel{kg}}{\cancel{kg} \frac{m^2}{\cancel{s^2}} \cancel{s^2}} = \frac{1}{m^2} \right|$$

$$= 1.62 \cdot \underline{10^{10}\ m^{-1}}$$

$\Downarrow$ $1/\kappa = 6.18 \cdot 10^{-11}\ m$

$\arcsin\sqrt{\frac{80}{81}} = \underline{1.46}$

2.

$$\Psi_n = \psi_n e^{-i\omega_n t}$$

$$\psi_n = A_n H_n(\xi) e^{-\xi^2/2}, \quad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x, \quad A_n = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}}$$

$$d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx$$

$$E_n = \hbar\omega(n + 1/2)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi_n^* x^2 \Psi_n \, dx = \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n \, dx =$$

$$= \int_{-\infty}^{\infty} A_n^2 H_n^2(\xi) e^{-\xi^2} x^2 \, dx = \left| \begin{array}{l} \xi = \sqrt{\frac{m\omega}{\hbar}}\, x \\ d\xi = \sqrt{\frac{m\omega}{\hbar}}\, dx \end{array} \right|$$

$$= \int_{-\infty}^{\infty} A_n^2 H_n^2(\xi) e^{-\xi^2} \frac{\xi^2}{\frac{m\omega}{\hbar}} \frac{d\xi}{\sqrt{\frac{m\omega}{\hbar}}}$$

$$= \left(\frac{\hbar}{m\omega}\right)^{3/2} A_n^2 \int_{-\infty}^{\infty} H_n^2(\xi) e^{-\xi^2} \xi^2 \, d\xi$$

$$= \left| \begin{array}{l} 2\xi H_n(\xi) - 2n H_{n-1}(\xi) = H_{n+1}(\xi) \\ \xi^2 H_n^2(\xi) = \dfrac{(H_{n+1}(\xi) + 2n H_{n-1}(\xi))^2}{4} \end{array} \right|$$

$$= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2} A_n^2 \Big[\int_{-\infty}^{\infty} H_{n+1}^2(\xi) e^{-\xi^2} d\xi + 4n \int_{-\infty}^{\infty} H_{n+1}(\xi) H_{n-1}(\xi) e^{-\xi^2} d\xi$$

$$+ 4n^2 \int_{-\infty}^{\infty} H_{n-1}^2(\xi) e^{-\xi^2} d\xi \Big] =$$

$$= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2} A_n^2 \left[2^{n+1}(n+1)!\sqrt{\pi} + 0 + 4n^2 \cdot 2^{n-1}(n-1)!\sqrt{\pi} \right]$$

$$= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2} A_n^2 \, 2^{n+1} n! \sqrt{\pi} \left[n+1+n \right]$$

$$= \frac{1}{4}\left(\frac{\hbar}{m\omega}\right)^{3/2} \sqrt{\frac{m\omega}{\pi\hbar}} \frac{1}{\cancel{2^n}\cancel{n!}} \cancel{2^{n+1}}\cancel{n!}\cancel{\sqrt{\pi}} \left[n+1+n \right]$$

$$= \frac{1}{2} \frac{\hbar}{m\omega} (2n+1) = \frac{\hbar}{m\omega} (n + 1/2)$$

$$E_n = m\omega^2 \langle x^2 \rangle$$

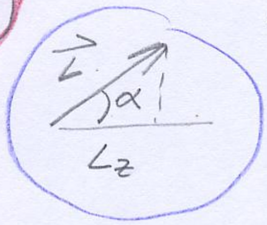
$$= \cancel{m} \omega^2 \frac{\hbar}{\cancel{m}\cancel{\omega}} (n + 1/2)$$

$$= \hbar\omega (n + 1/2)$$

3.

$\alpha_{min} = 24{,}1^\circ$

$L = ?$



$$\cos\alpha = \frac{L_z}{L} = \frac{m\hbar}{\sqrt{l(l+1)}\,\hbar}$$

$$\min\alpha \Rightarrow \max\cos\alpha \Rightarrow \max m = l$$

$$\cos(\min\alpha) = \frac{l}{\sqrt{l(l+1)}}$$

$\min\alpha = 24{,}1^\circ$

$L = ?$

$$\frac{l}{\sqrt{l(l+1)}} = \cos 24{,}1^\circ$$

$$\frac{l}{\sqrt{l(l+1)}} = 0{,}913 \quad /^2$$

$$l^2 = 0{,}913^2\, l(l+1)$$

$$l^2(1-0{,}913^2) = 0{,}913^2\, l$$

$$l = \frac{0{,}913^2}{1-0{,}913^2} \approx 5 \qquad (l \in \mathbb{N}\,!)$$

$$L = \hbar\sqrt{l(l+1)} = \sqrt{30}\,\hbar$$

$$\left(L = \hbar\sqrt{30} = 1{,}054\cdot 10^{-34}\,\mathrm{Js}\cdot\sqrt{30} = 5{,}773\cdot 10^{-34}\,\mathrm{Js}\right)$$

4.

$$R_{10}(r) = \left(\frac{1}{a_0}\right)^{3/2} 2 e^{-r/a_0}$$

$$P = \int \Psi^* \Psi \, dV$$

$$P = \int_{1.00 a_0}^{1.01 a_0} dr\, r^2 |R_{10}|^2 \overbrace{\int d\Omega\, Y_{10}^* Y_{10}}^{=1} =$$

$$= \left(\frac{1}{a_0}\right)^3 \cdot 4 \int_{1.00 a_0}^{1.01 a_0} dr\, r^2 e^{-2r/a_0} =$$

BRONŠTAJN, str. 443, rel. 449

$$\int x^2 e^{ax} dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

$$a = -\frac{2}{a_0}$$

$$= \frac{4}{a_0^3} \; e^{-\frac{2r}{a_0}}\left(\frac{r^2}{(-2/a_0)} - \frac{2r}{4/a_0^2} + \frac{2}{-8/a_0^3}\right)\Bigg|_{1.00 a_0}^{1.01 a_0}$$

$$= \frac{4}{a_0^3}\left[e^{-2\cdot 1.01}\left(-\frac{1.01^2 a_0^2 a_0}{2} - \frac{1.01 a_0 a_0^2}{2} - \frac{a_0^3}{4}\right)\right.$$

$$\left. - e^{-2}\left(-\frac{a_0^2 a_0}{2} - \frac{a_0 a_0^2}{2} - \frac{a_0^3}{4}\right)\right]$$

$$= 4\left[e^{-2.02}\left(-\frac{1.01^2}{2} - \frac{1.01}{2} - \frac{1}{4}\right)\right.$$

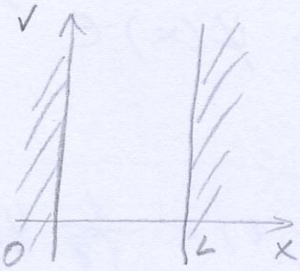
$$\left. - e^{-2}\left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{4}\right)\right]$$

$$= 4 \cdot 0.00135331$$

$$= 0.0054$$

1. T

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{inače} \end{cases}$$



a) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\underbrace{\frac{2m}{\hbar^2}E}_{k^2}\psi$$ odgovarajuća vremenski neovisna sch. jed.

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$ rješenje

rubni uvjeti: $\psi(0)=0$ $\quad A+B=0 \Rightarrow B=-A$

$\psi(L)=0$ $\quad A(e^{ikL}-e^{-ikL})=0$

$$2iA\underbrace{\left(\frac{e^{ikL}-e^{-ikL}}{2i}\right)}_{\sin kL}=0$$

$\sin kL = 0 \Rightarrow kL = n\pi$ $\quad n\in\mathbb{N}$

$$\Rightarrow k^2L^2 = n^2\pi^2$$

$$\frac{2m}{\hbar^2}EL^2 = n^2\pi^2$$

$$\Rightarrow \boxed{E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2}$$

$[\ \psi(x) = A'\sin kx\ ]$

$$\int_{-\infty}^{\infty}|\psi(x)|^2dx = 1 \Rightarrow \frac{1}{A'^2} = \int_0^L \sin^2 kx\, dx = \left|\text{Bronštajn, str. 427, rel. 275}\right|$$

$$= \left[\frac{1}{2}x - \frac{1}{4k}\sin 2kx\right]_0^L$$

$$= \frac{1}{2}L + 0 \Rightarrow A' = \sqrt{\frac{2}{L}}$$

$$\boxed{\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x}$$ $n\in\mathbb{N}$

b) $\left[\psi(x,t) = \psi(x)\, e^{-i/\hbar\, E_n t} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) e^{-i/\hbar\, E_n t} \right]$

c) $\psi(x,t) = \frac{1}{\sqrt{2}} \psi_2(x,t) + \frac{1}{\sqrt{2}} \psi_3(x,t)$

$\hat{H} \psi(x,t) = E\, \psi(x,t)$?

$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left[\frac{1}{\sqrt{2}} \psi_2(x,t) + \frac{1}{\sqrt{2}} \psi_3(x,t) \right]$

$= \frac{1}{\sqrt{2}} E_2 \psi_2(x,t) + \frac{1}{\sqrt{2}} E_3 \psi_3(x,t)$

$\neq cte\ \psi(x,t)$ $\Rightarrow$ nije svojstveno stanje energije

d) ne

gustoća vjerojatnosti
$\rho(x,t) = \psi^*(x,t)\, \psi(x,t) = \psi_2^2(x) + \psi_3^2(x) + \psi_2(x) \cdot \psi_3(x) \left(e^{i(E_2 - E_3)t/\hbar} + e^{-i(E_2-E_3)t/\hbar} \right)$

$\Rightarrow \rho(x,t) \neq \rho(x)$
tj. ovisi o vremenu → a vidjeno i iz c)

str. 86 LIBOFF!
"H eigenstates = stationary states"
!!

e) $\int_0^L \psi^*(x,t)\, \psi(x,t)\, dx$

$= \frac{1}{2} \int_0^L \left(\psi_2^*(x,t) + \psi_3^*(x,t) \right) \left(\psi_2(x,t) + \psi_3(x,t) \right) dx$

$= \frac{1}{2}(1+1) + \frac{1}{2} \underbrace{\int_0^L \sin\frac{2\pi x}{L} \sin\frac{3\pi x}{L}}_{=0} \left(e^{\frac{i}{\hbar}(E_2 - E_3)t} + e^{i/\hbar (E_3 - E_2)t} \right)$

$= 1$ ✓

$\Leftarrow \sin\alpha \cdot \sin\beta = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta) \right)$

BRONŠTAJN, str. 431, rel. 313

$\int \cos ax\, dx = \frac{1}{a} \sin ax$

g) da $\hat{\Pi}\, \psi(x,t) = -\psi(x,t)$

$\sin(-x) = -\sin x$

2. T

$$\varphi(\theta,\phi) = A(\sin\theta)^2 \sin 2\phi$$

a)

$$\varphi(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{|m|\le l} a_{lm} Y_l^m(\theta,\phi)$$

$$a_{lm} = \langle \varphi | Y_l^m \rangle = \int$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; A(\sin\theta)^2 \sin 2\phi \; Y_l^m(\theta,\phi)$$

$$\varphi(\theta,\phi) = A(\sin\theta)^2 \sin 2\phi$$

$$= A(\sin\theta)^2 \frac{e^{i2\phi} - e^{-i2\phi}}{2i}$$

$$= A \cdot 4\sqrt{\frac{2\pi}{15}} \left(Y_2^2 - Y_2^{-2}\right)\frac{1}{2i} = \frac{2A}{i}\sqrt{\frac{2\pi}{15}}\left(Y_2^2 - Y_2^{-2}\right)$$

$$a_{lm} = \begin{cases} -2Ai\sqrt{\frac{2\pi}{15}} & l=2 \quad m=-2 \\ 2Ai\sqrt{\frac{2\pi}{15}} & l=2 \quad m=-2 \end{cases}$$

b)

$$L^2 = l(l+1)\hbar^2 = 2\cdot 3\hbar^2 = 6\hbar^2 \quad \Rightarrow \quad P = a_{22}^2 + a_{2,-2}^2 = 1$$

$l=2$

$$L_z = m_l \hbar = \pm 2\hbar \quad \Rightarrow \quad P = a_{22}^2 = 1/2$$

$m=2$

$m=-2$ $\quad P = a_{2,-2}^2 = 1/2$

c)

$$\left|\pm 2Ai\sqrt{\frac{2\pi}{15}}\right|^2 = \frac{1}{2}$$

$$4A^2\frac{2\pi}{15} = \frac{1}{2} \quad \Rightarrow \quad A = \sqrt{\frac{15}{2\pi}}\frac{1}{\sqrt{8}} = \frac{1}{4}\sqrt{\frac{15}{\pi}}$$