

## Associated Laguerre Polynomials

Some wag once said the nice thing about standards is that there are so many to choose from. I have been trying to come to grips with the difference between what I presented in class and the formulae in Sakurai. It is easy to explain the differences on the basis of different conventions about the associated Laguerre polynomials.

If you want to skip details, a main result is that Sakurai and *Mathematica* use different conventions. If we call  $\mathcal{L}_n^q(\rho)$  the convention of Sakurai and  $L_p^{(q)}(\rho)$  the convention of *Mathematica*, we have

$$\mathcal{L}_{p+q}^q(\rho) = (p+q)!(-1)^q L_p^{(q)}(\rho) .$$

Below are the details. They are presented somewhat in the order of my investigation and not according to the shorted derivation of the above result.

### *Differential equation*

I have consulted two well known books on mathematical functions that adhere to the same index convention, but have different normalization conventions. The first book that I consulted by Abramowitz & Stegun states on pg 778, Eqs. (22.5.16) and (22.5.17):

$$L_n^{(0)}(x) = L_n(x)$$

$$L_n^{(m)}(x) = (-1)^m \frac{d^m}{dx^m} [L_{n+m}(x)]$$

Also, on pg 781, in Eq. (22.6.15), the differential equation is given.

$$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0 .$$

The differential equation is very valuable, but being linear, does not tell us anything about the normalization.

Another well known book by Morse & Feshbach on pg 784, in an unnumbered equation three lines from the bottom of the page gives their convention for the associated Laguerre polynomials.

$$L_n^m(z) = (-1)^m \frac{d^m}{dz^m} [L_{n+m}^0(z)] .$$

The differential equation is also given a few lines above:

$$z \frac{d^2}{dz^2} L_n^a(z) + (a + 1 - z) \frac{d}{dz} L_n^a(z) + n L_n^a(z) = 0 .$$

Morse & Feshbach do not put the upper index in parentheses, otherwise, it looks like these conventions might agree. We can be pretty certain that in these two books the  $L_n^{(a)}$  is a polynomial of degree  $n$ . However, we will soon see that the normalizations don't agree in the two books.

### *Sakurai convention*

Now, let's turn to Sakurai. On pg 454 in Eq. (A.6.4), we find

$$L_p^q(\rho) = \frac{d^q}{d\rho^q} L_p(\rho) .$$

This leads us to conclude that  $L_p^q$  is of degree  $p - q$ , and makes the result above plausible. In fact, if the normalizations were the same, we would expect:

$$\mathcal{L}_{p+q}^q(\rho) = \frac{d^q}{d\rho^q} L_{p+q}(\rho) = (-1)^q L_p^{(q)}(\rho) \quad \textbf{Not quite correct!} .$$

### *Class Derivation*

In class, I presented the differential equation for the associated Laguerre polynomials as stated by *Mathematica*,

$$xy'' + (a + 1 - x)y' + ny = 0 .$$

This is the same convention as Abramowitz & Stegun and Morse & Feshbach.

In class, we found we needed to solve this differential equation:

$$\rho L'' + (2(l + 1) - \rho)L' + (\lambda - l - 1)L = 0 ,$$

but  $\lambda = n$ , the total quantum number, and  $n - l - 1 = n'$  the radial quantum number. So, we have

$$\rho L'' + (2l + 1 + 1 - \rho)L' = n'L = 0 .$$

In the notation of Abramowitz & Stegun, *Mathematica* or the Morse & Feshbach index convention, the solution to the differential equation is

$$L_{n'}^{(2l+1)}(\rho) = L_{n-l-1}^{(2l+1)}(\rho) .$$

In Sakurai notation,  $L_{n-l-1}^{(2l+1)}(\rho) = (-1)^{2l+1} \mathcal{L}_{n+l}^{2l+1} = -\mathcal{L}_{n+l}^{2l+1}$  . This explains the indices for  $R_{nl}$  in Sakurai in the equation above (A.6.3).

### *Pinning Down the Normalizations*

We still need to consider normalization conventions, and that can be done from the generating function or from what is known as Rodrigues' formula. In fact, in retrospect, it seems that just looking at the Rodrigues' formulae in the three books might have been the easiest way to proceed.

In Abramowitz & Stegun, we find on pg 785, Eq. (22.11.6)

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}] .$$

On pg 784 of Morse & Feshbach, we find

$$L_n^a(z) = \frac{\Gamma(a+n+1)}{\Gamma(n+1)} \frac{e^z}{z^\alpha} \frac{d^n}{dz^n} [z^{a+n} e^{-z}] .$$

If we set  $\alpha$  and  $a$  to zero, we can compare with Sakurai, which states in Eq. (A.6.5)

$$L_p(\rho) = e^\rho \frac{d^p}{d\rho^p} (\rho^p e^{-\rho}) .$$

We immediately see that Sakurai agrees in normalization with Morse & Feshbach, at least for the Laguerre polynomials, if not for the associated Laguerre polynomials. However, the two books on mathematical methods differ by a factor of  $(n+a)!$  in their normalizations with Abramowitz & Stegun convention being smaller by division by that factor. Morse & Feshbach include a small table of associated Laguerre polynomials at the bottom of page 784. They have  $L_0^n = n!$ , whereas Abramowitz & Stegun according to Eq. (22.4.7) have  $L_0^{(\alpha)} = 1$ . The only remaining mystery is which normalization convention *Mathematica* obeys. With this command

Table[{n, LaguerreL[0, n, x]}, {n, 0, 6}]

you will easily find that all results are 1 and *Mathematica* follows the Abramowitz & Stegun normalization.

Further, I coded up the Rodrigues' formula with the Sakurai convention and compared with  $(p+q)!(-1)^q L_p^{(q)}$  where I used the *Mathematica* function `LaguerreL[p,q,x]`. They were in agreement.

Mystery solved! Quantum mechanics and children can now sleep soundly at night.